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On magnetoacoustic–gravity waves propagating or standing vertically in an atmosphere

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Abstract. Vertical hydromagnetic waves are considered in isothermal and non-isothermal atmospheres, with a constant external magnetic field, in the two cases where it is (i) vertical and (ii) horizontal, corresponding respectively to (i) a transversal Alfvèn–gravity wave and (ii) a longitudinal magnetosonic–gravity wave. It is shown that in any atmosphere (with bounded temperature and vanishing density at high altitude) the following laws hold for the asymptotic wavefields: (a) for propagating waves the velocity perturbation grows linearly and the magnetic field perturbation is asymptotically constant; (b) for standing modes the velocity perturbation is finite (but non-zero) and the magnetic field perturbation decays exponentially to zero. These properties contrast strongly with acoustic–gravity waves, which exhibit exponential growth with altitude, whether propagating or standing. The laws (a) and (b) are confirmed for isothermal atmospheres, in which case all asymptotic parameters can be calculated in terms of Bessel or hypergeometric functions, as particular forms of the expressions giving the velocity and magnetic field perturbations exactly at all altitudes and for all frequencies, both for standing modes and propagating waves. The magnetosonic–gravity wave evolves from a hydrodynamic regime similar to acoustic–gravity waves, through a transition layer, to a hydromagnetic regime similar to Alfvèn–gravity waves. The latter exhibits hydromagnetic behaviour at all altitudes, which is illustrated by plotting: (i) the waveforms for the first four standing modes; (ii) the amplitudes and phases of propagating waves of four different wavelengths. In both cases are included wavelengths comparable to or larger than the scale height, over which the atmospheric density and Alfvèn speed change substantially.

1. Introduction

Waves in fluids due to compressibility, magnetic fields and gravity are known respectively as acoustic (Rayleigh 1945, Morse and Ingard 1968), Alfvèn (1942, 1948) and internal (Rayleigh 1890, Yih 1965), and constitute a developing subject (Whitham 1974, Lighthill 1978). Three types of two-wave interactions are possible, namely acoustic–gravity (Lamb 1908, Campos 1983a), magnetogravity (Uchida 1968, Howe 1969) and magnetoacoustic (Lighthill 1960, Campos 1977) waves. The general three-wave interaction has been treated, using the WKBJ approximation (Lighthill 1964, Campos 1983c), by means of dispersion relations (McLellan and Winterberg 1968 Bray and Loughhead 1974) or by following the evolution of wavenumber surfaces with altitude (Lighthill 1967, Eltayeb 1977).

The need for exact (rather than approximate) solutions valid over the entire altitude range (not just locally) was appreciated fairly early (Ferraro 1954, Hide 1956), leading

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to the study of particular cases of hydromagnetic waves propagating or trapped in atmospheres, isothermal or non-isothermal (the latter with constant temperature gradient), with horizontal, vertical or oblique constant external magnetic fields (Meyer 1968, Zhugzhda 1971, Hollweg 1972, Thomas 1976, Nye and Thomas 1976, Yanowitch 1980).

Atmospheric phenomena occurring over many scale heights are dominated by vertical waves, which we will examine in several cases of magnetoacoustic–gravity modes (§ 1), including: (i) isothermal and non-isothermal atmospheres at rest, the latter with bounded temperature profile (§ 2); (ii) constant external magnetic fields, either horizontal or vertical, leading respectively to Alfvèn–gravity and magnetosonic–gravity waves (§ 3); (iii) standing modes reflected from infinity (§§ 4, 6) and upward or downward propagating waves (§§ 5, 7); (iv) plots of the waveforms, amplitudes and phases versus altitude for wavelengths comparable to or larger than the scale height (§ 8).

A fundamental aspect which may not have been sufficiently emphasised in the literature is the qualitative (not just quantitative) difference between the properties of (standing or propagating) waves in atmospheres with or without an external magnetic field (Campos 1983b). In the present instance this difference can be demonstrated by comparing: (i) the sound and Alfvèn speeds; (ii) the gas and magnetic pressures; (iii) the wave equations with magnetic field absent or present. The influence of these effects accumulates with altitude, and is most noticeable in the difference between the asymptotic fields of hydrodynamic and hydromagnetic waves.

The sound speed C_0 depends (only) on temperature (for a perfect gas), and is constant in an isothermal atmosphere, and finite in a non-isothermal atmosphere with bounded temperature. The Alfvèn speed C_1 varies inversely with (the square root of) density, and thus diverges at high altitude, as density tends to zero in any atmosphere, isothermal or not. The variation of the Alfvèn speed with altitude cannot reasonably be ignored, since it occurs over twice the density scale height (for a constant external magnetic field).

Assuming that the atmosphere is at rest and its properties do not depend on time, the wave frequency ω is conserved, and the acoustic wavelength $\lambda_0 = 2\pi C_0/\omega$ is finite whereas the Alfvèn wavelength $\lambda_1 = 2\pi C_1/\omega$ diverges with altitude. Thus we have the following contrast between the asymptotic fields of monochromatic waves: (i) the hydrodynamic wave retains a sinusoidal waveform in space and time; (ii) the hydromagnetic wave is sinusoidal only in time, and becomes monotonic in space at high altitude, e.g., the asymptotic amplitude is either constant or decays or grows steadily (without oscillation).

The gas pressure in an atmosphere is the weight of the fluid above, and thus decays to zero with altitude; the magnetic pressure is constant throughout the atmosphere if the external magnetic field is uniform. Thus, even if the waves are generated in an atmospheric region where the gas pressure dominates the magnetic pressure, as the wave propagates upward, the latter will take over above a transition layer. This is demonstrated by the three modes of vertical magnetoacoustic–gravity waves (figure 1): (i) the acoustic–gravity mode is a longitudinal hydrodynamic wave; (ii) the Alfvèn–gravity mode is a transverse hydromagnetic wave; (iii) the coupled magnetosonic–gravity mode has a transition from a hydrodynamic to a hydromagnetic regime.

The wave equation in the absence of an external magnetic field has constant coefficients (in an isothermal atmosphere for which the sound speed and scale height are constant), whereas in the presence of an external magnetic field the coefficients vary substantially (in the isothermal case the Alfvèn speed varies exponentially over twice the scale height). Thus we have the following contrast: (i) hydrodynamic waves are represented by complex exponentials, i.e., have sinusoidal waveforms, grow or decay exponentially and have linear phases; (ii) hydromagnetic waves are represented by special functions, e.g., Bessel (§§ 4, 5) or hypergeometric (§§ 6, 7), of regular (§§ 4, 6) or singular (§§ 5, 7) type, and may be *not* periodic in space, e.g., grow linearly, have bounded amplitude or decay, and have asymptotically finite phase.

Another consequence of hydromagnetic wave equations having variable coefficients is that the velocity and magnetic field perturbations satisfy different laws, namely, (i) for propagating waves (figure 2) the velocity perturbation grows linearly and the magnetic field perturbation is asymptotically constant, the phase (figure 3) being finite for both; (ii) for standing modes (figure 4) the velocity perturbation is asymptotically finite (but not zero) and the magnetic field perturbation decays exponentially. These asymptotic laws are proved generally for any atmosphere (with bounded temperature and vanishing density at high altitude), and are confirmed in the isothermal case, in which the wavefields are calculated exactly over the entire altitude range.

2. Magnetoacoustic–gravity wave operator

The general equations of compressible flow under magnetic $\boldsymbol{H}$ and gravity $\boldsymbol{g}$ fields are, in the absence of (viscous and resistive) dissipation,

$$\partial \boldsymbol{H}/\partial t + \nabla \times (\boldsymbol{H} \times \boldsymbol{v}) = 0, \qquad \partial \rho/\partial t + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{1a,b}$$

$$\rho[\partial \boldsymbol{v}/\partial t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}] + \nabla p = \rho \boldsymbol{g} - (\mu/4\pi)\boldsymbol{H} \times (\nabla \times \boldsymbol{H}), \tag{1c}$$

where ρ, p, $\boldsymbol{v}$ denote the density, pressure and velocity and μ the magnetic permeability. The induction equation (1a) states that the magnetic field lines are 'frozen in' the (perfectly conducting) fluid, the equation of continuity (1b) expresses conservation of mass, and the momentum equation (1c) balances the inertia force and pressure gradients against weight and magnetic force.

The flow quantities are assumed to be the superposition of: (i) an atmospheric mean state of rest, with a constant external magnetic field $\boldsymbol{H}_0$ of arbitrary direction, and pressure $p_0(z)$ and density $\rho_0(z)$ varying with altitude in agreement with hydrostatic equilibrium $\nabla p_0 = \rho_0 \boldsymbol{g}$; (ii) an unsteady and non-uniform perturbation of velocity $\boldsymbol{v}'$, pressure p', density ρ' and magnetic field $\boldsymbol{h}'$:

$$(\boldsymbol{V}, \boldsymbol{H}, p, \rho) = (\boldsymbol{0}, \boldsymbol{H}_0, p_0(z), \rho_0(z)) + (\boldsymbol{v}, \boldsymbol{h}, p', \rho'). \tag{2}$$

The atmosphere is generally not isentropic (this would correspond to neutral stability (e.g. Landau and Lifshitz 1953, § 4)), but we assume the propagation of perturbations to be adiabatic:

$$\partial p'/\partial t + v \cdot \nabla p' = C_0(\partial \rho'/\partial t + v \cdot \nabla \rho'), \qquad [C_0(z)]^2 \equiv (\partial p_0/\partial \rho_0)_s, \tag{3a,b}$$

where $C_0(z)$ denotes the sound speed and s the entropy density.

Considering waves of small amplitude with regard to the perturbations:

$$\partial \boldsymbol{h}/\partial t - (\boldsymbol{H}_0 \cdot \nabla)\boldsymbol{v} + \boldsymbol{H}_0(\nabla \cdot \boldsymbol{v}) = 0, \qquad \partial \rho'/\partial t + \boldsymbol{v} \cdot \nabla \rho_0 + \rho_0(\nabla \cdot \boldsymbol{v}) = 0, \tag{4a,b}$$

$$\rho_0 \partial \boldsymbol{v}/\partial t + \nabla p' = \rho' \boldsymbol{g} + (\mu/4\pi)[\nabla(\boldsymbol{H}_0 \cdot \boldsymbol{h}) - (\boldsymbol{H}_0 \cdot \nabla)\boldsymbol{h}], \tag{4c}$$

$$\partial p'/\partial t + \rho_0(\boldsymbol{v} \cdot \boldsymbol{g}) + \rho_0 C_0(\nabla \cdot \boldsymbol{v}) = 0. \tag{4d}$$

These linearised equations establish the following balances: (a) induction: magnetic field oscillation against velocity transport along mean magnetic field lines and fluid compressibility; (b) continuity: density oscillation against atmospheric stratification and fluid compressibility; (c) momentum: local inertia against perturbation pressure gradient, buoyancy force and cross-stresses between mean and perturbation magnetic fields; (d) adiabaticity: pressure oscillation against gravity and compressibility.

The system (4*a–d*) can be eliminated for the velocity $\boldsymbol{v}$, by applying $\partial/\partial t$ to (4*c*) and substituting $\partial \boldsymbol{h}/\partial t$, $\partial\rho'/\partial t$ and $\partial p'/\partial t$ respectively from (4*a, b, d*):

$$\partial^2 v/\partial t^2 - C_0^2 \nabla(\nabla\cdot\boldsymbol{v}) - \nabla(\boldsymbol{g}\cdot\boldsymbol{v}) - (\gamma-1)\boldsymbol{g}(\nabla\cdot\boldsymbol{v})$$
$$+ (\mu/4\pi\rho_0)[-(\boldsymbol{H}_0\cdot\nabla)\boldsymbol{v} + \boldsymbol{H}_0(\boldsymbol{H}_0\cdot\nabla)(\nabla\cdot\boldsymbol{v}) - H_0^2\nabla(\nabla\cdot\boldsymbol{v})$$
$$+ (\boldsymbol{H}_0\cdot\nabla)\nabla(\boldsymbol{H}_0\cdot\boldsymbol{v})] = 0. \tag{5}$$

Introducing the Alfvèn speed C_1 and the unit vector $\boldsymbol{l}$ along magnetic field lines,

$$[C_1(z)]^2 \equiv \mu H_0^2/4\pi\rho(z), \qquad \boldsymbol{l} \equiv \boldsymbol{H}_0/|\boldsymbol{H}_0|, \tag{6a, b}$$

equation (5) is written

$$\Box_{ij}(\partial/\partial t, \nabla, \boldsymbol{l}\cdot\nabla)v_j(\boldsymbol{x}, t) = 0, \tag{7a}$$

where the magnetoacoustic–gravity wave operator is given by

$$\Box_{ij} \equiv \delta_{ij}\,\partial^2/\partial t^2 - [C_1(z)]^2\,\partial^2/\partial x_i\,\partial x_j - g_j\,\partial/\partial x_i - (\gamma-1)g_i\,\partial/\partial x_j$$
$$-C_1(z)]^2(\delta_{ij}\,\partial^2/\partial l^2 - l_j\,\partial^2/\partial l\,\partial x_i) - [C_1(z)]^2(\partial^2/\partial x_i\,\partial x_j - l_i\,\partial^2/\partial l\,\partial x_j), \tag{7b}$$

and involves, besides the derivative with regard to time, $\partial/\partial t$, the spatial derivatives: isotropic, $\partial/\partial x_i$, and along magnetic field lines, $\partial/\partial l \equiv \boldsymbol{l}\cdot\nabla \equiv l_j\,\partial/\partial x_j$.

The deduction of equation (5) which has been presented holds for non-isothermal atmospheres under a constant external magnetic field (McLellan and Winterberg 1968, Bray and Loughhead 1974, p 251), and in the form (7*b*) it brings together the magnetoacoustic (Campos 1977) and acoustic–gravity (Campos 1983a) wave operators, since it includes the following terms (from left to right): (i) second-order time dependence, allowing waves propagating in opposite directions and their superposition into standing modes; (ii) isotropic non-dispersive acoustic waves involving the dilatation $\nabla\cdot\boldsymbol{v} \equiv \partial v_j/\partial x_j$; (iii) anisotropic, dispersive internal waves involving the acceleration of gravity g_j; (iv) acoustic–gravity coupling through the gravitational acceleration and dilatation; (v)–(vi) non-dispersive, one-dimensional Alfvèn–gravity waves propagating along the direction $\boldsymbol{l}$ of the external magnetic field, with a stratification effect on the Alfvèn speed $C_1(z)$; (vii)–(viii) magnetoacoustic coupling through the dilatation $\partial v_j/\partial x_j$ and Alfvèn speed $C_1(z)$, the stratification effect being again present in the latter.

3. Asymptotic wavefields in non-isothermal atmosphere

Oblique hydromagnetic waves in atmospheres can be reflected or tunnelled (Chiu 1971), and phenomena occurring over many scale heights are dominated by vertical waves, which depend only on altitude z and time t. Whereas three-dimensional waves are described clearly by the magnetoacoustic–gravity wave equation (5), for vertical waves it is more convenient to start from the perturbation equations (4*a*–*d*) considering two cases: (i) if the external magnetic field is vertical, $\boldsymbol{H}_0 \equiv (0, 0, H_1)$, the only propagating components of the velocity and magnetic field perturbations are v_x, h_x, which satisfy

$$\partial h_x/\partial t = H_1\, \partial v_x/\partial z, \qquad \partial v_x/\partial t = H_1^{-1} C_1^2\, \partial h_x/\partial z; \tag{8a, b}$$

(ii) if the external magnetic field is horizontal (and the x axis aligned with it), $\boldsymbol{H}_0 \equiv (H_{11}, 0, 0)$, the propagating components are v_z, h_x, and satisfy

$$\partial h_x/\partial t + H_{11}\, \partial v_z/\partial z = 0, \tag{9a}$$

$$\partial^2 v_z/\partial t^2 - C_0^2\, \partial^2 v_z/\partial z^2 + \gamma g\, \partial v_z/\partial z + H_{11}^{-1} C_1^2\, \partial^2 h_x/\partial t\, \partial z = 0. \tag{9b}$$

Thus we have three modes of vertical magnetoacoustic–gravity waves:

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14187 **Table 1.** Comparison of magnetoacoustic–gravity wave modes.

Mode	Slow	Fast	Alfvèn
Designation	Acoustic–gravity	Magnetosonic–gravity	Alfvèn–gravity
Type	Longitudinal	Coupled	Transversal
Dynamics	Compressible	Compressible	Incompressible
Magnetics	Amagnetic	Magnetic	Magnetic
Diagram			
Altitude z (vertically upwards)			

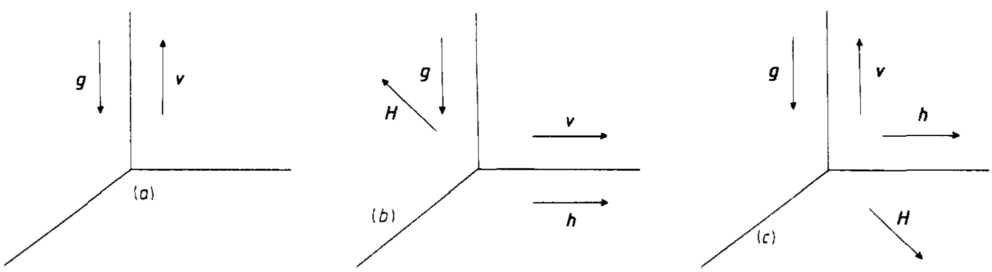

Figure 1. Sketches of the three magnetoacoustic–gravity wave modes in the case of vertical propagation. (*a*) Acoustic–gravity wave; (*b*) Alfvèn–gravity wave; (*c*) magnetosonic–gravity wave.

Mode	Slow	Fast	Alfvèn
Atmosphere			
Gravity $\boldsymbol{g}$	$(0, 0, -g)$	$(0, 0, -g)$	$(0, 0, -g)$
Magnetic field $\boldsymbol{H}$	$(0, 0, 0)$	$(H_{11}, H_y, 0)$	$(0, H_y, H_1)$
Density	$\rho(z)$	$\rho(z)$	$\rho(z)$
Pressure	$p(z)$	$p(z)$	$p(z)$
Perturbation			
Velocity $\boldsymbol{v}(z, t)$	$(0, 0, v_z)$	$(0, 0, v_z)$	$(v_x, 0, 0)$
Density $\rho'(z, t)$	$\rho' \neq 0$	$\rho' \neq 0$	$\rho' = 0$
Magnetic field $\boldsymbol{h}(z, t)$	$(0, 0, 0)$	$(h_x, 0, 0)$	$(h_x, 0, 0)$
Dissipation			
Dynamic viscosity	$\nu \neq 0$	$\nu \neq 0$	$\nu \neq 0$
Ohmic conductivity	$\sigma = \infty$	$\sigma \neq \infty$	$\sigma \neq \infty$
Hall conductivity	$\zeta = \infty$	$\zeta \neq \infty$	$\zeta = \infty$

(i) the acoustic–gravity mode is independent of the magnetic field ($C_1^2=0$ in (9b)),

$$\{\partial^2/\partial t^2-[C_0(z)]^2\,\partial^2/\partial z^2+\gamma g\,\partial/\partial z\}v_z(z,t)=0, \qquad (10)$$

and is longitudinal (the velocity perturbation lies in the direction of propagation), and hence propagates a dilatation $\nabla\cdot\boldsymbol{v}=\partial v_z/\partial z$; (ii) the Alfvèn–gravity mode (8a, b),

$$\{\partial^2/\partial t^2-[C_1(z)]^2\,\partial^2/\partial z^2\}v_x(z,t)=0, \qquad (11)$$

corresponds to a vertical external magnetic field, and propagates velocity and magnetic field perturbations which are transversal and parallel; (iii) the magnetosonic–gravity mode (9a, b),

$$[\![\partial^2/\partial t^2-\{[C_0(z)]^2+[C_1(z)]^2\}\,\partial^2/\partial z^2+\gamma g\,\partial/\partial z]\!]v_z(z,t)=0, \qquad (12)$$

corresponds to a horizontal external magnetic field, and propagates a longitudinal velocity and transverse magnetic field perturbation, thus coupling compressibility with magnetism.

Eliminating (8a, b) for the magnetic field perturbation we obtain: for Alfvèn–gravity waves

$$\{\partial^2/\partial t^2-\partial/\partial z[C_1(z)]^2\,\partial/\partial z\}h_x(z,t)=0, \qquad (13a)$$

and for magnetosonic–gravity waves

$$[\![\partial^2/\partial t^2-\partial/\partial z\{[C_0(z)]^2+[C_1(z)]^2\}\,\partial/\partial z+\gamma g\,\partial/\partial z]\!]h_x(z,t)=0. \qquad (13b)$$

Comparison of (13a) with (11) shows that for Alfvèn–gravity waves the velocity and magnetic field perturbations satisfy: (i) the same equation (11) for a homogeneous medium, for which the Alfvèn speed (6a) is constant; (ii) equations differing by the term (which appears in (13a) in addition to those in (11)) $2C_1(\mathrm{d}C_1/\mathrm{d}z)\partial/\partial z$, which is non-zero in a non-uniform medium, such as an atmosphere. A comparison of (13b) with (12) would lead to similar conclusions for magnetosonic–gravity waves, with the extra term (in (13b), not appearing in (12)) being $2(C_0\,\mathrm{d}C_0/\mathrm{d}z+C_1\,\mathrm{d}C_1/\mathrm{d}z)\partial/\partial z$, where the acoustic part is zero only in an isothermal atmosphere, whereas the Alfvèn part is not zero even in the isothermal case. The difference in waveform for the velocity V and magnetic field H perturbation spectra is demonstrated by the induction equations (8a), (9a), which for a monochromatic wave of frequency ω (i.e. factor $\mathrm{e}^{-\mathrm{i}\omega t}$) read respectively for Alfvèn–gravity and magnetosonic–gravity waves

$$H(z;\omega)=\mathrm{i}(H_1,-H_{11})\omega^{-1}\,\mathrm{d}V(z;\omega)/\mathrm{d}z, \qquad (14a, b)$$

and allow the magnetic field perturbation to be calculated from the velocity one without solving the second wave equation (13a, b).

A standing hydromagnetic wave has a velocity perturbation which is finite (but not necessarily zero) at high altitude, and thus, by (14a, b), an asymptotically vanishing magnetic field perturbation:

$$v(z,t)\sim V(0;\omega)d(\omega)\,\mathrm{e}^{-\mathrm{i}\omega t}, \qquad h(z,t)\sim 0, \qquad (15a, b)$$

where $d(\omega)$ is the constant ratio of asymptotic to initial velocity, which generally depends on frequency ω. For propagating waves we note that (11), (12) imply

$$\mathrm{d}^2V(z;\omega)/\mathrm{d}z^2\sim \mathrm{O}(\omega^2C_1^{-2}V),\ \mathrm{O}\{\omega^2C_1^{-2}[1+(C_0/C_1)^2]^{-1}V\}, \qquad (16a, b)$$

respectively for Alfvèn–gravity and magnetosonic–gravity waves. In any atmosphere the density decays with altitude, i.e. $\rho(z)\to 0$ as $z\to\infty$, and thus the Alfvèn speed diverges, $C_1 \to \infty$, so that the RHS of (13*a*, *b*) vanishes (the sound speed C_0 remains bounded if the temperature is finite and hence $C_0/C_1 \to 0$). From $d^2V/dz^2 \sim 0$ it follows that the velocity perturbation grows linearly, and from (14*a*, *b*) that the magnetic field perturbation is asymptotically constant:

$$V(z;\omega) \sim V(0;z)[a(\omega)z + b(\omega)], \qquad H(z;\omega) \sim V(0;\omega)(H_1, -H_{11})\omega^{-1}a(\omega). \tag{17a, b}$$

The asymptotic laws (15*a*, *b*), (17*a*, *b*) have been proved generally for any atmosphere with asymptotically vanishing density (and bounded temperature), and will be confirmed in the isothermal case, for which the functions a, b, $d(\omega)$ can be calculated explicitly.

4. Standing Alfvèn–gravity modes

In an isothermal atmosphere the density $\rho(z) = \rho_0\,e^{-z/L}$ decays exponentially over the scale height L, and thus the Alfvèn speed (6*a*) increases over twice the scale height:

$$C_1(z) = c_1\,e^{z/2L}, \qquad c_1^2 \equiv \mu H_1^2/4\pi\rho_0, \tag{18a, b}$$

where c_1 denotes the Alfvèn speed at altitude $z = 0$. Since the atmospheric properties do not depend on time, we may use a Fourier decomposition:

$$v_x, h_x(z,t) = \int_{-\infty}^{+\infty} V, H(z;\omega)\,e^{-i\omega t}\,d\omega, \tag{19a, b}$$

where V, H denote the velocity and magnetic field perturbation spectra, and the frequency ω is conserved.

Substituting (18*a*), (19*a*) in (11), we obtain the equation for the velocity perturbation spectrum of vertical Alfvèn–gravity waves in an isothermal atmosphere:

$$[d^2/dz^2 + (\omega/c_1)^2\,e^{-z/L}]V(z;\omega) = 0. \tag{20a}$$

The magnetic field perturbation spectrum satisfies a different equation (13),

$$[d^2/dz^2 + L^{-1}\,d/dz + (\omega/c_1)^2\,e^{-z/L}]H(z;\omega) = 0, \tag{20b}$$

and is related to the velocity perturbation by (14*a*).

The problem involves only one dimensionless quantity, namely, the scattering parameter

$$\alpha = \omega L/C_1 = k_1L = 2\pi L/\lambda_1, \qquad \Delta \equiv \rho_0/\rho(\lambda_1) = e^{\lambda_1/L} = e^{2\pi/\alpha}, \tag{21a, b}$$

where k_1 is the wavenumber, λ_1 the wavelength and Δ the atmospheric density change over a wavelength. Since the atmospheric properties vary with altitude, the wavelength and wavenumber are not conserved, the quantities indicated being local values (i.e. the wavelength and wavenumber in a homogeneous medium whose density would equal the local density). The WKBJ approximation $\alpha^2 \equiv k_1^2L^2 \gg 1$ corresponds to a small atmospheric density change $\Delta \sim 1$ over a wavelength, whereas the exact theory is required for long waves $k_1L \sim 1$, for which the atmospheric density changes substantially, $\Delta \gg 1$, over a wavelength.

In order to solve (20*a*) exactly we perform a change of variable:

$$u \equiv 2\alpha\,e^{-z/2L} = (2\omega L/c_1)\,e^{-z/2L}, \qquad d/dz = -(u/2L)\,d/du, \tag{22a, b}$$

transforms the operator (20a) into $u\,\mathrm{d}^2/\mathrm{d}u^2+\mathrm{d}/\mathrm{d}u+u$, which is that of the Bessel equation of order zero; thus we can write the solution of (20a) as a linear combination of Bessel functions of order zero (and first and second kinds) with variable u (22a):

$$V(z\,;\omega)=AJ_0[(2\omega L/c_1)\mathrm{e}^{-z/2L}]+BY_0[(2\omega L/c_1)\,\mathrm{e}^{-z/2L}], \tag{23}$$

where A, B are arbitrary constants of integration.

If we consider waves perfectly reflected from the atmospheric layer $z=z_0$, the corresponding upper boundary condition requires the vanishing of the velocity perturbation $V(z_0;\,\omega)=0$. In the limit $z_0\to\infty$ when the reflecting layer recedes to infinity, we cannot require a vanishing velocity perturbation, since as $z\to\infty$ and $u\to 0$ (by (22a)), the Bessel functions satisfy $J_0(u)\to 1$ and $Y_0(u)\to\infty$, and thus (23) cannot vanish asymptotically (except in the trivial case $A=0=B$), since the first term is bounded and non-zero and the second diverges. We can have an asymptotically finite but non-zero velocity perturbation by setting $B=0\neq A$ in (23), to obtain a 'node at infinity'. In this case the kinetic energy (per unit volume) $\rho v^2/2\sim \mathrm{O}(\mathrm{e}^{-z/L})$ decays like the atmospheric density, and the magnetic energy $\mu h^2/8\pi\sim\mathrm{O}(\mathrm{e}^{-2z/L})$ decays with altitude at a faster rate, since for asymptotically bounded velocity it will be shown (see (28b)) that the magnetic field perturbation decays like $h\sim\mathrm{O}(\mathrm{e}^{-z/L})$. Thus the total energy per unit volume E, which is the sum of the kinetic and magnetic energies, vanishes asymptotically, $E\sim\mathrm{O}(\mathrm{e}^{-z/L})$, showing that no energy is radiated to or comes from infinity. This confirms that the condition $B=0$ selects in (23) a standing wave, and the remaining constant of integration can be determined from the initial velocity perturbation spectrum:

$$V_0(\omega)\equiv V(0;\omega)=AJ_0(2\omega L/c_1), \tag{24}$$

so that the velocity perturbation is given, at altitude z and time t, by (19a):

$$v_x(z,t)=\int_{-\infty}^{+\infty} V_0(\omega)\{J_0[(2\omega L/c_1)\,\mathrm{e}^{-z/2L}]/J_0(2\omega L/c_1)\}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega. \tag{25}$$

If we denote by a_n the zeros of the Bessel function of the first kind of order zero $J_0(a_n)=0$, which are simple, then the integrand of (25) has simple poles for

$$\omega_n=c_1a_n/2L, \qquad \lambda_n\equiv 2\pi c_1/\omega_n=4\pi L/a_n, \tag{26a, b}$$

which are respectively the frequencies and (reference) wavelengths of standing vertical Alfvèn–gravity modes in an isothermal atmosphere. The compactness parameter (21a) for the nth mode, $\alpha_n=a_n/2\equiv k_nL=2\pi L/\lambda_n=a_n/2$, shows that the WKBJ approximation would not apply to the first few standing modes, since the density change $\Delta_n\equiv\exp(\lambda_n/L)=\exp(4\pi/a_n)$ is substantial over a wavelength.

The integral (25) can be evaluated as $\pi\mathrm{i}$ times the sum of the residues at the simple poles on the positive real axis for which $n=1,2,\dots$ and $\omega_n>0$ (excluding $n=-1,-2,\dots$ and $\omega_n<0$), and denoting by $b_n\equiv -J_0'(a_n)=J_1(a_n)$ the slope of the Bessel function J_0 at its zero a_n, we obtain for the velocity perturbation

$$v_x(z,t)=(\pi c_1/2L)\sum_{n=1}^{\infty}\mathrm{Im}[V_0(\omega_n)\,\mathrm{e}^{-\mathrm{i}\omega_n t}]b_n^{-1}J_0(a_n\,\mathrm{e}^{-z/2L}), \tag{27a}$$

and from (8a) for the magnetic field perturbation

$$h_x(z,t)=(\pi H_1/2L)\,\mathrm{e}^{-z/2L}\sum_{n=1}^{\infty}\mathrm{Re}[V_0(\omega_n)\,\mathrm{e}^{-\mathrm{i}\omega_n t}]b_n^{-1}J_1(a_n\,\mathrm{e}^{-z/2L}). \tag{27b}$$

Thus the wavefield is represented at all altitudes and times, for both the velocity (27*a*) and magnetic field (27*b*) perturbation, as a superposition of the standing modes (26*a*).

The expressions (27*a*, *b*) remain valid at high altitude as $z\to\infty$, $u\to 0$, $J_0(u)\to 1$ and $J_1(u)\to u/4$, and thus the velocity and magnetic field perturbations for Alfvèn–gravity waves standing vertically in an isothermal atmosphere are given by

$$v_x(z,t)\sim(\pi c_1/2L)\sum_{n=1}^{\infty} b_n^{-1}\,\mathrm{Im}[V_0(\omega_n)\,\mathrm{e}^{-\mathrm{i}\omega_n t}], \tag{28a}$$

$$h_x(z,t)\sim(\pi H_1/8L)\,\mathrm{e}^{-z/L}\sum_{n=1}^{\infty}(a_n/b_n)\,\mathrm{Re}[V_0(\omega_n)\,\mathrm{e}^{-\mathrm{i}\omega_n t}], \tag{28b}$$

to an exponential order of approximation $\mathrm{O}[(a_n^2/16)\,\mathrm{e}^{-z/L}]$, the same accuracy being obtained at lower altitude for lower-order modes.

The results (28*a*, *b*) agree with the general predictions (15*a*, *b*) that standing hydromagnetic waves have bounded velocity perturbation and exponentially decaying magnetic field perturbation, and specify the function $d(\omega)$ for the nth Alfvèn–gravity mode in an isothermal atmosphere:

$$\mathrm{e}^{\mathrm{i}\omega_n t}v_x(\infty;t)/V_0(\omega_n)\equiv d(\omega_n)=-\mathrm{i}\pi c_1/2Lb_n=-\mathrm{i}\pi c_1\omega_n/a_nb_n. \tag{29}$$

Thus the growth in the asymptotic velocity perturbation relative to the initial value is larger for higher-order modes.

5. Propagating Alfvèn–gravity waves

The solution (23) of (20*a*) can also be written as a linear combination of Hankel functions of (order zero and) first and second kinds:

$$V(z;\omega)=AH_0^{(1)}[(2\omega L/c_1)\,\mathrm{e}^{-z/2L}]+BH_0^{(2)}[(2\omega L/c_1)\,\mathrm{e}^{-z/2L}], \tag{30}$$

where A, B are new constants of integration. The Hankel functions have asymptotic forms (Watson 1944, p 198)

$$H_0^{(1,2)}(u)\sim(2/\pi u)^{1/2}\exp[\pm\mathrm{i}(u-\pi/4)][1+\ \ (u^{-1})], \tag{31}$$

and thus $H^{(1)}(H^{(2)})$ correspond respectively to waves propagating in the direction of increasing (decreasing) u; since by (22*a*) the increasing (decreasing) u correspond to decreasing (increasing) altitude z, the first (second) term of (30) should represent respectively a downward (upward) propagating wave.

This conclusion can be checked if we consider high-frequency waves for which the WKBJ approximation $\lambda_1^2/L^2\ll 1$ applies over a fraction of the scale height $z^2\ll L^2$, so that the overall change in the wave is small. This corresponds to large u (22*a*):

$$u=2\omega L/c_1-\omega z/c_1+\mathrm{O}(\omega z^2/c_1L)=2k_1L-k_1z+\mathrm{O}(z^2/\lambda_1L), \tag{32a}$$

for which the asymptotic form (31) yields

$$H^{(1,2)}(u)=(c_1/\pi\omega L)^{1/2}\,\mathrm{e}^{z/4L}\,\mathrm{e}^{\mp k_1z}\,\mathrm{e}^{\pm(2k_1L-\pi/4)}[1+\mathrm{O}(\lambda_1/L)], \tag{32b}$$

showing that the first (second) term of (30) has the factor $\mathrm{e}^{-\mathrm{i}k_1z}$ ($\mathrm{e}^{+\mathrm{i}k_1z}$) corresponding respectively to a downward (upward) propagating wave. From (32*b*) it follows that in both cases the amplitude of the high-frequency Alfvèn–gravity wave grows initially with altitude over four times the scale height $v\sim\mathrm{e}^{z/4L}$; this result agrees with the constancy of the initial energy flux $J \sim \rho v^2 C_1$, bearing in mind that the density decays like $\rho \sim O(e^{-z/L})$ and the Alfvèn speed (18*a*) grows like $C_1 \sim O(e^{z/2L})$.

It will be noted that the two terms of (30) cannot be readily identified as propagating waves, since the sinusoidal waveform $\exp(\pm ik_1 z)$ is never assumed by low-frequency waves, and even for very high-frequency waves the accumulated effects of stratification change the sinusoidal waveform into an unrecognisable shape within a few scale heights. Thus, in order to identify with certainty the wavefields arising from the wave initially propagating upward, we have followed the following procedure (which, of course, also applies to downward propagating waves): (i) we apply the *radiation condition* (Lighthill 1964), selecting only the upward propagating term with factor $\exp(ik_1 z)$, in the high-frequency, low-altitude (WKBJ) approximation; (ii) we retain, for all frequencies and at all altitudes, only that term which complies with the radiation condition in the WKBJ limit, and corresponds to a wave emitted upwards at level $z = 0$. The procedure (ii) could be designated *consistency principle* in analogy (Lighthill, private suggestion) with the correspondence principle of quantum mechanics (Landau and Lifshitz 1966, § 6), which requires that in the short-wave limit $\lambda \to 0$ the classical solution in a homogeneous medium be regained.

Thus, to select an upward propagating wave we set $A = 0$ in (30), and determine the remaining constant of integration from the initial velocity perturbation spectrum:

$$V_0(\omega) \equiv V(0;\omega) = BH_0^{(2)}(2\omega L/c_1). \tag{33}$$

Since the Hankel function does not vanish for real argument there are no resonant modes, and (30) can be divided by (33) to yield the velocity perturbation spectrum

$$V(z;\omega) = [V_0(\omega)/H_0^{(2)}(2\omega L/c_1)]H_0^{(2)}[(2\omega L/c_1)\,e^{-z/2L}], \tag{34a}$$

and the magnetic field perturbation spectrum (14*a*)

$$H(z;\omega) = i(H_1/c_1)\,e^{-z/2L}[V_0(\omega)/H_0^{(2)}(2\omega L/c_1)]H_1^{(2)}[(2\omega L/c_1)\,e^{-z/2L}], \tag{34b}$$

for an Alfvèn–gravity wave propagating vertically in an isothermal atmosphere.

The Hankel function $H^{(2)}$ has a logarithmic singularity as $u \to 0$ (Kamke 1971, vol 1, p 439):

$$H_0^{(2)}(u) = -i(2/\pi)\log u + 1 - i2\phi/\pi + O(u^2/4), \tag{35}$$

(where ϕ denotes Euler's constant) which corresponds to the linear divergence of the velocity perturbation spectrum with altitude,

$$V(z;\omega) \sim V_0(\omega)[H_0^{(2)}(2\omega L/c_1)]^{-1}[(i/\pi L)z + 1 - i2\phi/\pi - i(2/\pi)\log(2\omega L/c_1)], \tag{36a}$$

and asymptotic constancy of the magnetic field perturbation spectrum,

$$H(z;\omega) \sim -(H_1/\pi\omega)V_0(\omega)[H_0^{(2)}(2\omega L/c_1)]^{-1}. \tag{36b}$$

The asymptotic laws (36*a*, *b*) are valid to a high order of approximation $O[(k_1^2 L^2/4)\,e^{-z/L}]$, the same accuracy being obtained at lower altitude for longer waves.

The results (36*a*, *b*) agree with the general property of propagating hydromagnetic waves (17*a*, *b*) having linearly diverging velocity and asymptotically constant magnetic field perturbations, and specify the functions $a(\omega)$, $b(\omega)$ for Alfvèn–gravity waves in an isothermal atmosphere:

$$a(\omega), b(\omega) = [H_0^{(2)}(2\omega L/c_1)]^{-1}\{i/\pi L,\ 1 - i2\phi/\pi - i(2/\pi)\log(2\omega L/c_1)\}. \tag{37a, b}$$

The function $a(\omega)$, in particular, specifies both the rate of growth of the velocity perturbations and the asymptotic value of the magnetic field:

$$a(\omega) \sim [V_0(\omega)]^{-1}\, dV(z;\omega)/dz = -i(\omega/H_1)[V_0(\omega)]^{-1}H(z;\omega). \tag{38a, b}$$

Concerning the energy unit volume: (i) although the velocity perturbation diverges linearly, $v \sim O(z)$, the kinetic energy $\rho v^2/2 \sim O(z^2\, e^{-z/L})$ vanishes asymptotically at high altitude; (ii) since the magnetic field perturbation h is asymptotically constant, so is the magnetic energy $\mu h^2/8\pi$. Thus the situation is asymptotically similar to a plane propagating Alfvèn wave as concerns the magnetic energy per unit volume which is constant, but not for the kinetic energy which vanishes. Thus, in contrast with the equipartition of kinetic and magnetic energy for Alfvèn waves propagating in a homogeneous medium, for Alfvèn–gravity waves propagating upward in an atmosphere: (i) there is equipartition of energy only initially, namely, $H \sim (H_1/c_1)V$ for $z/L \ll 1$ in (34a, b) leading to $\rho V^2/2 \sim (\rho/2)(c_1/H_1)^2 H^2 \sim \mu H^2/8\pi$; (ii) as the wave propagates upward into more rarefied atmospheric regions, the magnetic predominates over the kinetic energy, and asymptotically all energy is magnetic. This conclusion, which follows from the asymptotic laws (17a, b), applies to Alfvèn–gravity waves propagating in isothermal or non-isothermal atmospheres with bounded temperature.

6. Standing magnetosonic–gravity modes

The acoustic (3b) and Alfvèn (6a) speeds are given respectively by

$$[C_0(z)]^2 = \gamma p(z)/\rho(z), \qquad [C_1(z)]^2 = 2P/\rho(z), \tag{39a, b}$$

where $p(z)$ denotes the gas pressure and P the magnetic pressure. The analogy between the formulae (39a, b) is completed if we note that for a perfect gas $\gamma = 1 + 2/N$, where N is the number of (rotational and translational) degrees of freedom of the molecule, so that we have the following cases: (i) for a three-dimensional, polyatomic molecule $N = 6$, $\gamma = \frac{4}{3}$; (ii) for a diatomic gas (or linear polyatomic molecule) $N = 5$, $\gamma = \frac{7}{5}$; (iii) for a monatomic gas $N = 3$, $\gamma = \frac{5}{3}$; (iv) the value $\gamma = 2$ used in (39b) corresponds to $N = 2$, i.e. a magnetic gas whose molecules have only two degrees of freedom, transverse to the direction of propagation (which coincides with magnetic field lines for Alfvèn waves).

The ratio of sound and Alfvèn speeds (squared) is also the ratio of gas magnetic pressures:

$$[C_0(z)/C_1(z)]^2 = (\gamma/2)[p(z)/P], \tag{40}$$

apart from a constant factor $\gamma/2 = \frac{2}{3}, \frac{7}{10}, \frac{5}{6}$ respectively for polyatomic (i), diatomic (ii) or monatomic (iii) perfect gases. Thus the magnetosonic–gravity wave equation (12) describes the transition between two regimes: (i) the hydrodynamic regime in regions sufficiently dense for the gas pressure to dominate the magnetic pressure,

$$(\partial^2/\partial t^2 - C_0^2\, \partial^2/\partial z^2 + C_0^2 L^{-1}\, \partial/\partial z)v_z(z,t) = O[(2P/\gamma p)\, \partial^2 v_z/\partial z^2], \tag{41}$$

corresponds to the vertical acoustic–gravity wave operator (Moore and Spiegel 1964, Campos 1983a); (ii) the hydromagnetic regime, at altitudes sufficiently high for the gas pressure to be negligible compared with the magnetic pressure, corresponds to the Alfvèn–gravity wave equation (11) for $v_z(z,t)$ to order $O[(\gamma p/2P)\, \partial^2 v_z/\partial z^2]$.

The transition from the hydrodynamic to the hydromagnetic regime as the wave propagates upward, and the influence of the initial acoustic propagation on the magnetically dominated asymptotic wavefield, will be examined by solving (12) exactly in an isothermal atmosphere, for which the sound speed c_0 is constant and the Alfvèn speed is given by ((18a, b) with H_{11}):

$$[\partial^2/\partial t^2 - (c_0^2 + c_1^2\, e^{z/L})\, \partial^2/\partial z^2 + c_0^2 L^{-1}\, \partial/\partial z]v_z(z, t) = 0, \tag{42a}$$

or, for the velocity perturbation spectrum (19a),

$$\{[1 + (c_1/c_0)^2\, e^{z/L}]\, d^2/dz^2 - L^{-1}\, d/dz - (\omega/c_0)^2\}V(z\,;\,\omega) = 0, \tag{42b}$$

which involves two dimensionless quantities: (i) the acoustic compactness parameter $\alpha \equiv \omega L/c_0$ (analogous to (21a)); (ii) the ratio of phase speeds c_1/c_0 at altitude $z = 0$.

If we perform in (42b) a change of variable,

$$u = -(c_0/c_1)^2\, e^{-z/L}, \qquad d/dz = L^{-1}\, d/du, \tag{43a, b}$$

the differential operator (42b) transforms into a hypergeometric one $(1-u)u\ d^2/du^2 + (1-2u)\, d/du - \omega^2 L^2/c_0^2$, with parameters (Forsyth 1929, p 214) $c = 1$, $a + b = 1$, $ab = \omega^2 L^2/c_0^2$. Thus, a, b are the roots of

$$0 = \upsilon^{-2}(a + b) - ab = \upsilon^2 - \upsilon - \omega^2 L^2/c_0^2, \tag{44}$$

and the solution of (42b) is a linear combination of hypergeometric functions of first and second kinds:

$$V(z\,;\,\omega) = AF(a, b\,; 1; -(c_0/c_1)^2\, e^{-z/L}) + BG(a, b; 1; -(c_0/c_1)^2\, e^{-z/L}), \tag{45}$$

where the coefficients A, B are arbitrary constants of integration.

The equation (44) has a double root for

$$\omega_* = c_0/2L, \qquad \lambda_* = 2\pi\omega_*/c_0 = 4\pi L, \tag{46a, b}$$

which are designated respectively the cut-off frequency (46a) and wavelength (46b), since: (i) below the cut-off frequency $\omega < \omega_*$ (or above the cut-off wavelength $\lambda > \lambda_*$)

$$a, b = (1 \pm \beta)/2, \qquad \beta \equiv |1 - \omega^2/\omega_*^2|^{1/2}, \tag{47a, b}$$

all quantities in (45) are real and only standing modes exist; (ii) above the cut-off frequency $\omega > \omega_*$ (or below the cut-off wavelength $\lambda < \lambda_*$)

$$a, b = \tfrac{1}{2} \pm iKL, \qquad K \equiv (\omega/c_0)|\omega^2/\omega_*^2 - 1|^{1/2} \tag{48a, b}$$

and since (45) involves complex quantities, propagating waves are possible.

It will be noted that the cut-off conditions (46)–(48a, b) are exactly the same as those for vertically propagating acoustic–gravity waves (Lamb 1932, p 542, Campos 1983a). These extend unchanged to magnetosonic–gravity waves since the magnetic field has no wave filtering properties, as demonstrated by the absence of a cut-off frequency for Alfvèn–gravity waves (§ 4). Thus the properties of the atmosphere as a high-pass filter of acoustic–gravity waves are inherited by the magnetosonic–gravity waves.

The hypergeometric function of the second kind is singular as $z \to \infty$ and $u \to 0$, and since the velocity perturbation of standing modes must be bounded (§ 4), we set $B = 0$ in (45). The remaining constant of integration is determined from the initial velocity perturbation spectrum:

$$V_0(\omega) \equiv V(0; z) = AF(1/2+\beta/2, -1/2-\beta/2; 1; -c_0^2/c_1^2). \tag{49}$$

Resonance occurs ($A = \infty$) whenever the atmosphere is excited ($V_0(\omega) \neq 0$) at frequencies ω_n which are roots of

$$0 = F(a, b; 1; -c_0^2/c_1^2)$$
$$= 1 + \sum_{p=1}^{\infty} [(-)^p (c_0/c_1)^{2p}/p!(p+1)!] \prod_{q=0}^{p-1} (q^2+q+\omega^2 L^2/c_0^2), \tag{50}$$

which defines the normal modes. These correspond to poles of the integrand of ((19a); (45) with $B = 0$; (49))

$$V_z(z, t) = \int_{-\infty}^{+\infty} V_0(\omega)[F(a, b; 1; -(c_0/c_1)^2 \mathrm{e}^{-z/L})/F(a, b; 1; -c_0^2/c_1^2)] \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}\omega. \tag{51}$$

This formula can be evaluated by residues to yield the velocity perturbation spectrum,

$$v_z(z, t) = \pi \sum_{\omega_n < \omega_*} \mathrm{Im}[V_0(\omega_n) \mathrm{e}^{-\mathrm{i}\omega_n t}]$$
$$\times f_n^{-1} F(1/2+\beta_n/2, 1/2-\beta_n/2; 1; -(c_0/c_1)^2 \mathrm{e}^{-z/L}), \tag{51a}$$

and the magnetic field perturbation spectrum (9a),

$$h_x(z, t) = (\pi H_{11} L/c_1^2) \mathrm{e}^{-z/L} \sum_{\omega_n < \omega_*} \mathrm{Re}[V_0(\omega_n) \mathrm{e}^{-\mathrm{i}\omega_n t}]$$
$$\times (\omega_n/f_n) F(3/2+\beta_n/2, 3/2-\beta_n/2; 2; -(c_0/c_1)^2 \mathrm{e}^{-z/L}), \tag{51b}$$

where (47b) $\beta_n = |1-(\omega_n/\omega_*)^2|^{1/2}$ and

$$f_n \equiv \partial F(1/2+\beta_n/2, 1/2-\beta_n/2; 1; -c_0^2/c_1^2)/\partial \omega_n. \tag{52}$$

The general wavefields (51a, b) of magnetosonic–gravity waves vertically standing in an isothermal atmosphere are thus a superposition of the standing modes whose frequencies (50) lie below the cut-off: $0 < \omega_n < \omega_*$.

The formulae (51a, b) remain valid at high altitude as $z \to \infty$ and the hypergeometric functions of vanishing argument tend to unity, yielding the asymptotic velocity and magnetic field perturbations for a magnetosonic–gravity wave standing in an isothermal atmosphere:

$$v_z(z, t) \sim \pi \sum_{\omega_n < \omega_*} f_n^{-1} \mathrm{Im}[V_0(\omega_n) \mathrm{e}^{-\mathrm{i}\omega_n t}], \tag{53a}$$

$$h_x(z, t) \sim (\pi H_{11} L/C_1^2) \mathrm{e}^{-z/L} \sum_{\omega_n < \omega_*} (\omega_n/f_n) \mathrm{Re}[V_0(\omega_n) \mathrm{e}^{-\mathrm{i}\omega_n t}], \tag{53b}$$

these formulae being valid to an exponential order of accuracy $\mathrm{O}[(1-\beta_n)^2 (c_0/c_1)^2 \mathrm{e}^{-z/L}]$. The results (53$a$, b) agree with the general prediction that standing hydromagnetic waves have bounded velocity perturbation (15a) and exponentially decaying magnetic field perturbation (15b), and specify the function

$$\mathrm{e}^{\mathrm{i}\omega_n t} v_z(\infty; t)/V_0(\omega_n) \equiv d(\omega_n) = -\mathrm{i}\pi/f_n. \tag{54}$$

for the nth mode in an isothermal atmosphere.

7. Propagating magnetosonic–gravity waves

For propagating waves a, b are given by (48a), and the velocity perturbation spectrum is given by (45),

$$V(z;\omega)=AF(1/2+\mathrm{i}KL, 1/2-\mathrm{i}KL; 1; -(c_0/c_1)^2\,\mathrm{e}^{-z/L})$$
$$+BG(1/2+\mathrm{i}KL, 1/2-\mathrm{i}KL; 1; -(c_0/c_1)^2\,\mathrm{e}^{-z/L}), \qquad (55)$$

as a linear combination of hypergeometric functions of first and second kinds, whose coefficients A, B are arbitrary constants of integration. The magnetic field perturbation spectrum follows from (14b):

$$H(z;\omega)=-\mathrm{i}(\omega LH_{11}/c_1^2)\,\mathrm{e}^{-z/L}[AF(3/2+\mathrm{i}KL, 3/L-\mathrm{i}KL; 2, -(c_0/c_1)^2\,\mathrm{e}^{-z/L})$$
$$+BG(3/2+\mathrm{i}KL, 3/2-\mathrm{i}KL; 2; -(c_0/c_1)^2\,\mathrm{e}^{-z/L})], \qquad (56)$$

where we have used the derivation formula $\mathrm{d}F(a,b;c;z)/\mathrm{d}z=(ab/c)\times F(a+1,b+1;c+1;z)$.

The formulae for the perturbations of magnetosonic–gravity waves propagating vertically in an isothermal atmosphere hold if $-1<u<+1$ in (43a), that is: (i) over the entire altitude range $0\leqslant z<\infty$, if at the bottom of the atmosphere the gas pressure is sufficiently low or the magnetic field strong enough for the Alfvèn speed to exceed the sound speed $c_1>c_0$; (ii) if, as is physically more common in wave generation regions $c_0>c_1$, the expressions (55a, b) are valid in the high-altitude range $z_*<z<\infty$, where the Alfvèn speed (18a) exceeds the sound speed:

$$z_*\equiv 2L\log(c_0/c_1)=L\log(\gamma p/2P). \qquad (57)$$

Thus the solution (55), (56) represents the hydromagnetic regime in regions where the magnetic pressure dominates the gas pressure.

The solution below the transition layer (57) is obtained by analytic continuation from u to $1/u$, using the formulae (Caratheodory 1964, vol 2, pp 168–9):

$$F(a,b;c;u)$$
$$=[\Gamma(c)\Gamma(b-a)/\Gamma(c-a)\Gamma(b)](-u)^{-a}F(a+1-c,a;a+1-b;1/u)$$
$$+\text{symmetric in }(a,b), \qquad (58a)$$

$$G(a,b;c;u)$$
$$=[\Gamma(b-a)/\Gamma(b)\Gamma(1-a)][2\phi-\psi(1-a)-\psi(b)+\pi\mathrm{i}](-u)^{-a}$$
$$\times F(a+1-c,a;a+1-b;1/u)+\text{symmetric in }(a,b), \qquad (58b)$$

where Γ, ψ denote respectively the gamma and psi functions and ϕ is Euler's constant. The velocity perturbation spectrum is given by (56):

$$V(z;\omega)=\Gamma(-2\mathrm{i}KL)[\Gamma(1/2-\mathrm{i}KL)]^{-2}(c_0/c_1)^{1-2\mathrm{i}KL}\,\mathrm{e}^{z/2L}\,\mathrm{e}^{\mathrm{i}Kz}$$
$$\times\{A+[2\phi-2\psi(1/2-\mathrm{i}KL)+\pi\mathrm{i}]B\}F(-(c_1/c_0)^2\,\mathrm{e}^{z/L})$$
$$+(\text{exchange }+K\text{ and }-K), \qquad (59)$$

where

$$F, G(u) \equiv F, G(1/2+iKL, 1/2+iKL; 1+2iKL; u),$$
$$F_+, G_+ \equiv F, G(3/2+iKL, 3/2+iKL; 2+2iKL; u), \quad (60a, b)$$

in the low-altitude range $0 \leqslant z < z_*$.

The constants of integration A, B can now be determined by applying two conditions. (i) The principle of consistency requires that we keep in (59) only that term which (in the WKBJ limit) meets the radiation condition and corresponds to an upward propagating wave; thus we retain the first term, which includes the factor e^{iKz}, and omit the second term (which would have the factor e^{-iKz}) by setting its coefficient equal to zero:

$$0 = A + [2\phi - 2\psi(1/2+iKL) + \pi i]B. \quad (61a)$$

(ii) The initial velocity perturbation spectrum:

$$V_0(\omega) \equiv V(0;\omega) = \Gamma(-2iKL)[\Gamma(1/2-iKL)]^{-2}(c_0/c_1)^{1-2iKL}$$
$$\times\{A + [2\phi - 2\psi(1/2-iKL) + \pi i]B\}F(-c_1^2/c_0^2)$$
$$= \sqrt{\pi}2^{2iKL}(c_0/c_1)^{1-2iKL}\tanh(\pi KL)[\Gamma(iKL)/\Gamma(1/2+iKL)]BF(-c_1^2/c_0^2), \quad (61b)$$

using (61a) and known properties of the gamma function.

Since the complex hypergeometric function in (61b) does not vanish for real ω, resonant modes ($V_0(\omega) \neq 0$, $B = \infty$) do not exist, and the spectrum is continuous above the cut-off frequency (46a). The quantity K appearing in (59), (60a, b), as well as in (55), (56), is the vertical wavenumber defined by (48b), which coincides with the ordinary wavenumber $K \sim \omega/c_0$ (in the WKBJ approximation) at high frequency ($\omega^2 \gg \omega_*^2$), reduces compared with $\omega/c_0 > K$ at intermediate frequencies ($\omega > \omega_*$), and vanishes at the cut-off ($K = 0$ for $\omega = \omega_*$) when propagation becomes impossible. From (59) and (61a) we obtain the velocity and magnetic field (14b) perturbation spectra:

$$V(z;\omega) = V_0(\omega)\, e^{z/2L}\, e^{iKz}[F(-(c_1/c_0)^2 e^{z/L})/F(-c_1^2/c_0^2)], \quad (62a)$$

$$H(z;\omega) = -iH_{11}[(1/2+iKL)/\omega L]V_0(\omega)\, e^{z/2L}\, e^{iKz}[F(-c_1^2/c_0^2)]^{-1}$$
$$\times[F(-(c_1/c_0)^2 e^{z/L}) - (c_1^2/2c_0^2)F_+(-(c_1/c_0)^2 e^{z/L})], \quad (62b)$$

valid in the low-altitude range.

Noting that $u < 0$ in (43a), and performing analytic continuation in the variable $0 < (1-u)^{-1} < 1$, by means of (Abramowitz and Stegun 1964, p 559)

$$F(a, b; c; 1/u) = (1-1/u)^{-a}F(a, c-b; c; 1/(1-u)), \quad (63)$$

we obtain from (62a, b) the following formulae for the velocity and magnetic field perturbation spectra:

$$V(z;\omega) = V_0(\omega)\, e^{z/2L}\, e^{iKz}[1+(c_1/c_0)^2 e^{z/I}]^{-1/2-iKL}$$
$$\times\{F([1+(c_0/c_1)^2 e^{-z/L}]^{-1})/F(-c_1^2/c_0^2)\}, \quad (64a)$$

$$H(z;\omega) = -iH_{11}[(1/2+iKL)/\omega L]V_0(\omega)\, e^{z/2L}\, e^{iKz}[F(-c_1^2/c_0^2)]^{-1}$$
$$\times[1+(c_1/c_0)^2 e^{-z/L}]^{-1/2-iKL}\{F[1+(c_0/c_1)^2 e^{-z/L}]$$
$$-(c_1^2/2c_0^2)[1+(c_1/c_0)^2 e^{-z/L}]^{-1}F_+([1+(c_0/c_1)^2 e^{-z/L}]^{-1})\}, \quad (64b)$$

which are valid over the whole altitude range $0 \leqslant z < \infty$, the notation (60a, b) being used.

Summarising, we have obtained expressions for the velocity and magnetic field spectrum of a magnetosonic–gravity wave propagating vertically in an isothermal atmosphere, which are valid in: (i) the low-altitude range $0 \leqslant z < z_*$ where (62a, b), the hydrodynamic regime, applies, in which the exponential growth of acoustic–gravity waves is modified by the magnetic terms in the hypergeometric functions; (ii) the high-altitude range $z_* < z < \infty$ where (55), (56), the hydromagnetic regime, applies, which will be similar to Alfvèn–gravity waves with a modification due to compressibility; (iii) the formulae (64a, b) valid over the entire altitude range $0 \leqslant z < \infty$ describe the transition from the hydrodynamic to the hydromagnetic regime, and show that the wavefield is finite at the transition layer $z = z_*$ (57) between the two regimes:

$$V(z_*; \omega) = V_0(\omega) 2^{-1/2 - iKL} (c_0/c_1)^{1+2iKL} [F(-c_1^2/c_0^2)]^{-1} F(1/2), \tag{65a}$$

$$H(z_*; \omega) = -iH_{11}[(1/2 + iKL)/\omega L] V_0(\omega) 2^{-1/2-iKL} (c_0/c_1)^{1+2iKL} \times [F(-c_1^2/c_0^2)]^{-1} [F(1/2) - (c_1/2c_0)^2 F_+(1/2)]. \tag{65b}$$

The asymptotic limit $z \to \infty$ corresponds (43a) to $u \to 0$, and hence to $F(u) = 1 + O(u)$ and $G(u) = \log uF(u) + O(u) = \log u + O(u, u \log u)$, so that the logarithmic singularity in the high-altitude formula (55) corresponds to the following asymptotic wavefields:

$$V(z; \omega) \sim [A + (\pi i + 2 \log(c_0/c_1))B] - B(z/L), H(z; \omega) \sim -i(H_{11}/\omega L)B, \tag{66a, b}$$

where A, B are given by (61a, b). These formulae are valid to a high order of approximation $O[(\omega L/c_1)^2 e^{-z/L}]$, and confirm the general prediction (17a, b) that hydromagnetic waves have linearly diverging velocity, and asymptotically constant magnetic field perturbations. The functions $a(\omega), b(\omega)$ are given by

$$a(\omega), b(\omega) = +\pi^{-1/2} 2^{-2iKL} (c_0/c_1)^{2iKL-1} [\Gamma(1/2 + iKL)/\Gamma(iKL)] \tag{67a}$$

$$\times \coth(\pi KL) \times [V_0(\omega)/F(-c_1^2/c_0^2)] \times [-1/L, -2\psi(1/2 + iKL) + 2 \log(c_0/c_1) + 2\phi], \tag{67b}$$

for magnetosonic–gravity waves propagating vertically in an isothermal atmosphere. These asymptotic laws imply that the kinetic $\rho v^2/2 \sim O(z^2 e^{-z/L})$ and compression $\rho'^2 c_0^2 / 2\rho \sim O(e^{-z/L})$ energies per unit volume vanish asymptotically, whereas the magnetic energy $\mu h^2/8\pi \sim O(1)$ is asymptotically constant, and thus at high altitude all energy is magnetic. Considering the energies integrated over an infinite column of fluid from $z = 0$ to $z = \infty$, the total kinetic and compression energies are finite (since $z^2 e^{-z/L}$ and $e^{-z/L}$ are integrable from 0 to ∞), but the total magnetic energy is infinite (as the total energy of a plane wave in an infinite column is also infinite).

A solution with finite magnetic energy over an infinite column of fluid from $z = 0$ to $z = \infty$ would be specified (for a given frequency ω) by the condition

$$E_g \equiv (\mu/8\pi) \int_0^\infty |H(z; \omega)|^2 \, dz = (\mu H_0^2/8\pi\omega^2) \int_0^\infty |dV(z; \omega)/dz|^2 \, dz < \infty, \tag{68}$$

where we have used (14a, b) and H_0 denotes the external magnetic field, vertical for Alfvèn–gravity and horizontal for magnetosonic–gravity waves. A condition analogous to (68) has been applied to viscous acoustic–gravity waves (Yanowitch 1967), and in the present case of non-dissipative magnetosonic–gravity waves (68) could be met by setting $B = 0$ in (56), i.e. choosing the hypergeometric function of the first kind alone, so that: (i) below the cut-off frequency $\omega < \omega_*$ we would obtain standing modes just as in § 6, where the condition $B = 0$ was indeed used, since standing waves must have a finite total energy even in an 'infinite cavity' corresponding to perfect reflections at $z = 0$ and $z = \infty$; (ii) above the cut-off frequency $\omega > \omega_*$, if we had used the condition (68) instead of the radiation condition (and consistency principle), then the propagating wavefields would be different from those in § 7, since the constants of integration in (55) would be given by $B = 0$ and $A = V_0(\omega)$ instead of (61*a*, *b*). It would follow that the low-altitude solution (62*a*) would be replaced by the sum of an upward propagating and a downward propagating (or reflected) wave, and the transition layer $z = z_*$ (57) between the low- and high-altitude ranges would become a reflecting layer, giving rise to the downward propagating wave, and implying that the energy density would vanish asymptotically.

A condition of the type (68) is met by standing modes, and it is also suited to propagating waves in the presence of dissipation, e.g., for viscous acoustic–gravity waves it requires a finite rate of dissipation by viscosity (Yanowitch 1967), and for resistive Alfvèn–gravity waves the condition of finite rate of Joule dissipation can be put into a similar form (Campos 1983b), essentially that the square of the modulus of dV/dz is integrable from 0 to ∞. This condition is *not* suited to propagating *non*-dissipative waves, e.g., it is *not* satisfied by a plane wave, and for this reason we have not used condition (68) in §§ 5, 7 which are concerned with the propagation of non-dissipative magnetoacoustic–gravity waves.

8. Waveforms, amplitudes and phases

When studying radiation and propagation phenomena it is usually appropriate (e.g. Campos 1978) to illustrate some of the points which have been made analytically by plotting the wavefields in a number of cases. The magnetosonic–gravity wave evolves from a modified hydrodynamic waveform through a transition region to a hydromagnetic one. Since the hydrodynamic waveforms have been illustrated elsewhere (Campos 1983a), we concentrate here on the hydromagnetic modes, taking as an illustration the simpler case of Alfvèn–gravity waves. For propagating waves we represent the ratio of the (velocity and magnetic field perturbation) spectra at altitude z to the initial spectra:

$$\mathscr{V} \equiv V(z;\omega)/V_0(\omega) = |V(z;\omega)/V_0(\omega)| \exp\{i \arg[V(z;\omega)/V_0(\omega)]\}, \tag{69a}$$

$$\begin{aligned}\mathscr{H} &\equiv i\omega L H(z;\omega)/V_0(\omega)H_{11} \\ &= (\omega L/H_{11})|H(z;\omega)/V_0(\omega)| \exp\{i \arg[V(z;\omega)/V_0(\omega)] + i\pi/2\}.\end{aligned} \tag{69b}$$

by plotting against (dimensionless) altitude z/L (divided by scale height L) the following two quantities: (i) the modulus or ratio of amplitude at altitude z to initial amplitude (at altitude $z = 0$) (figure 2); (ii) the argument, or phase difference accumulated from the level $z = 0$ where waves arise to the altitude z (figure 3).

The vertical propagation of Alfvèn–gravity waves in an isothermal atmosphere is characterised by a single parameter, the compactness α (21*a*) which specifies the ratio of wavelength to scale height $\lambda_1/L = 2\pi/\alpha$ and the density change Δ (21*b*) within a scale height. We give the compactness parameter four values, and record in the table

(70a) the following (36a, b) asymptotic values: (i) the dimensionless slope $(L/V_0)\,\mathrm{d}V/\mathrm{d}z$ of the velocity perturbation spectrum; (ii) the dimensionless magnitude of the magnetic field perturbation spectrum $\mathscr{H}$ (69b); (iii), (iv) the phase difference accumulated asymptotically over the altitude range $0<z<\infty$, for both velocity (arg $(\mathscr{V})$) and magnetic field (arg $(\mathscr{H})$) perturbation spectra:

$\alpha \equiv k_1 L$	0.5	1.0	2.0	5.0
$\lambda_1/L \equiv 2\pi/\alpha$	12.56	6.28	3.14	1.26
$\Delta \equiv \rho(0)/\rho(\lambda_1)$	2.87×10^5	5.35×10^2	2.31×10	3.51
$(L/V_0)\,\mathrm{d}V/\mathrm{d}z$	0.413	0.515	0.801	1.262
$\mathscr{H}$	0.826	0.515	0.400	0.252
arg $(\mathscr{V})$	−96.6°	−158.8°	177.6°	−257.2°
arg $(\mathscr{H})$	−6.6°	−68.8°	87.6°	−167.2°

(70a)

From the table (70a) or from figures 2 and 3 we can draw the following conclusions: (i) the four cases considered range from very long wavelengths over which the atmosphere density changes by several orders of magnitude to a case of gradual density change over a wavelength, thus including cases for which an exact theory is necessary and for which the WKBJ approximation would suffice locally; (ii) the velocity perturbation exhibits a linear growth with altitude above about three scale heights, the slope increasing with frequency; (iii) the magnetic field perturbation exhibits a constant value above an altitude of about three scale heights, the actual magnitude decreasing with frequency; (iv) the phase shift accumulated during propagation over the entire

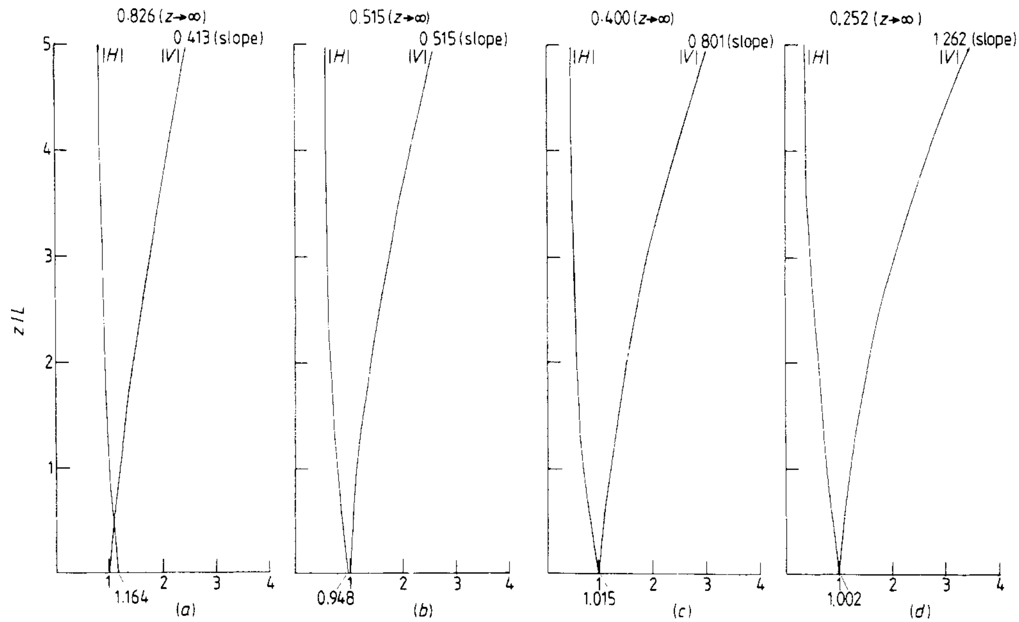

Figure 2. Alfvèn–gravity waves propagating vertically in an isothermal atmosphere: ratio of amplitudes versus altitude z/L for velocity (V) and magnetic field (H) perturbations, for four values of scattering parameter. (a) $kL=\frac{1}{2}$, (b) $kL=1.0$, (c) $kL=2.0$, (d) $kL=5.0$.

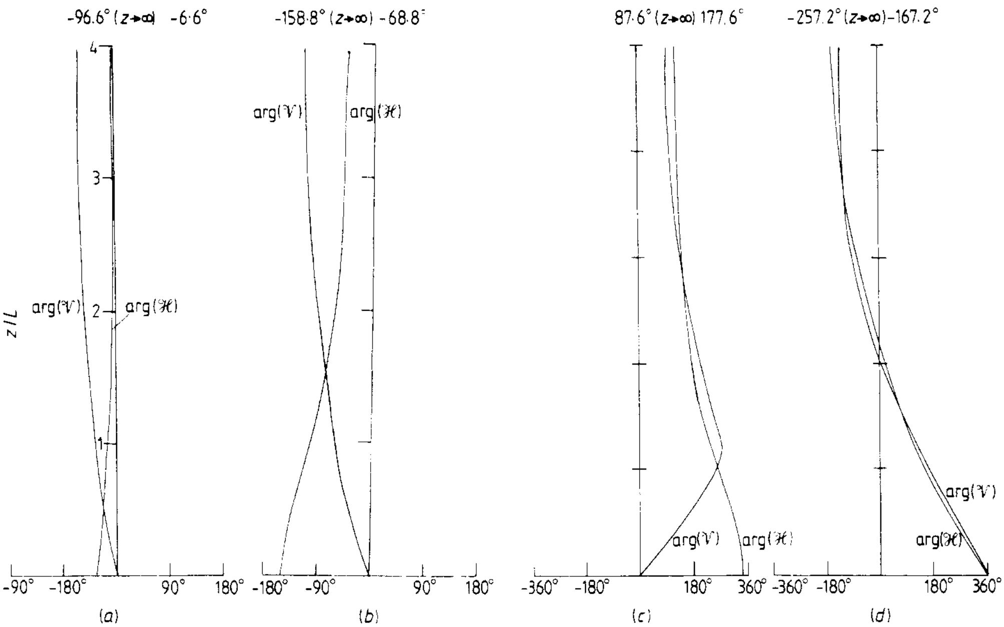

Figure 3. As for figure 2 except that the ordinate is the phase shift between altitudes z and 0. (a) $kL = 0.5$, (b) $kL = 1.0$, (c) $kL = 2$, (d) $kL = 5$.

atmosphere $0 < z < \infty$ is finite, and corresponds to the phase difference over the first four scale heights, the latter being of the same sign for the velocity and magnetic field perturbations, of larger modulus for the former, and increasing with frequency in both cases.

The modes of Alfvèn–gravity waves standing vertically in an isothermal atmosphere form a discrete spectrum, with compactness parameter $\alpha_n = a_n/2$ and ratio of wavelength to scale height $\lambda_n/L = 4\pi/a_n$ specified by (26b) the roots a_n of the Bessel function (of first kind, order zero) J_0. The magnetic field perturbation decays exponentially to zero, but velocity perturbation tends to a finite, non-zero value (29), both having $(n-1)$ nodes for the nth mode. The table below summarises these quantities for the first four modes.

n	1	2	3	4
$\alpha_n \equiv k_n L$	1.202	2.760	4.237	5.896
λ_n/L	5.225	2.276	1.452	1.060
$\Delta_n \equiv \rho(0)/\rho(\lambda_n)$	1.86×10^2	9.74	4.27	2.90
$V(\infty; \omega_n)/V_0(\omega_n)$	−1.926	2.938	−3.684	4.302
$V(z; \omega_n) = 0$ for $z/L =$	—	1.693	0.929 2.422	0.594 1.525 3.182
$H(z; \omega_n) = 0$ for $z/L =$	—	0.822	0.457 1.404	0.285 1.076 2.273

(70b)

From the table (70*b*) or figure 4 we can draw the following conclusions: (i) the first few standing modes correspond to long waves, with atmospheric density varying substantially over a single wavelength, so that the WKBJ approximation is inapplicable; (ii) the magnetic field perturbation decays exponentially and becomes negligible above a number of scale heights which increases with the order of the mode, from about two for the first mode, to about four scale heights for the fourth mode; (iii) the velocity perturbation tends to a constant asymptotic value, which increases with the order of the mode, and is reached after an increasing number of scale heights, e.g., two for the first mode and four for the fourth mode; (iv) the reason for the asymptotic law to take a larger number of scale heights to establish itself for higher-order modes is that the nth mode has $(n-1)$ nodes, which are interlaced with the $(n-2)$ nodes of the $(n-1)$th mode, the nodes of the velocity perturbation being always at higher altitude than the corresponding nodes of the magnetic field perturbation.

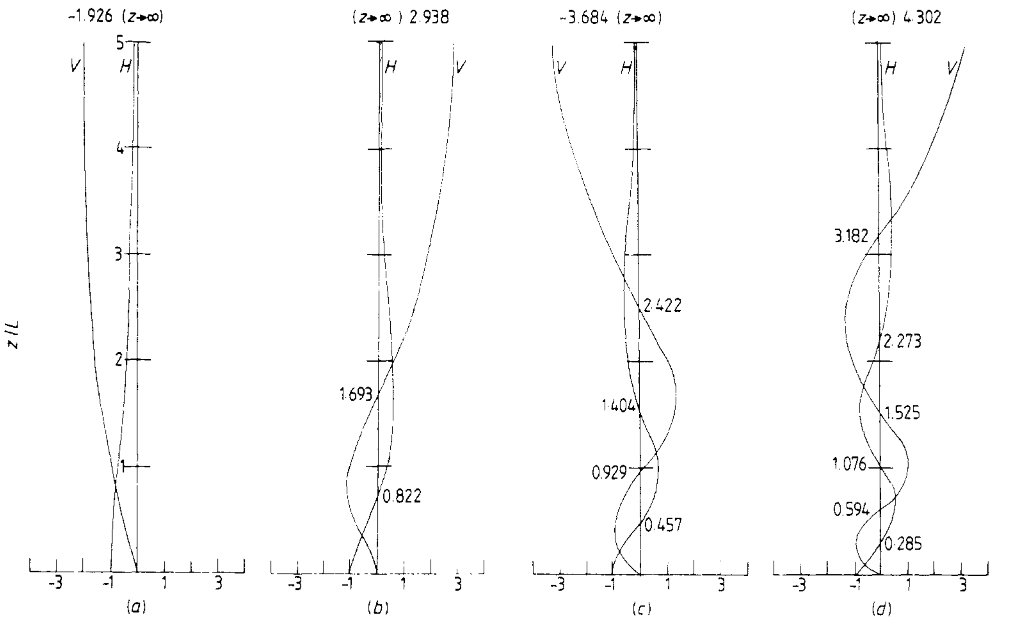

Figure 4. Waveforms of velocity (V) and magnetic field (H) perturbations plotted against dimensionless altitude z/L for the first four standing modes of Alfvèn-gravity waves perfectly reflected from infinity in an isothermal atmosphere. (a) $n=1$, (b) $n=2$, (c) $n=3$, (d) $n=4$.

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